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Definite and indefinite inner products on superspace (Hilbert–Krein superspace)

Florin Constantinescu

Fachbereich Mathematik, Johann Wolfgang Goethe-Universität Frankfurt,
Robert-Mayer-Strasse 10, D 60054 Frankfurt am Main, Germany

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Abstract
We present natural (invariant) definite and indefinite scalar products on the $N = 1$ superspace which turns out to carry an inherent Hilbert–Krein structure. We are motivated by supersymmetry in physics but prefer a general mathematical framework.

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1. Introduction

Supersymmetries generalize the notion of a Lie algebra to include algebraic systems whose defining relations involve commutators as well as anticommutators. Denoting by $Q_\alpha, \bar{Q}_{\dot\alpha}$ the odd (anticommuting) generators, physical considerations require that (see [1]) the operators $Q_\alpha, \bar{Q}_{\dot\alpha} = (Q_\alpha)^+$ act in a bona fide Hilbert space $\mathcal{H}$ of states with positive definite metric. Here $(Q_\alpha)^+$ means the operator adjoint to Q_α in $\mathcal{H}$. From the commutation relations [1]

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^l_{\alpha\dot\alpha} P_l$$

where σ^l, $l = 0, 1, 2, 3$ are the Pauli matrices with $\sigma^0 = -1$ as in [1] and P_l is the momentum, it follows that for any state Φ in $\mathcal{H}$ we have

$$\|Q_\alpha\Phi\|^2 + \|\bar{Q}_{\dot\alpha}\Phi\|^2 = (\Phi, \{Q_\alpha, \bar{Q}_{\dot\alpha}\}\Phi) = 2\sigma^l_{\alpha\dot\alpha}(\Phi, P_l\Phi).$$

Summing over $\alpha = \dot\alpha = 1, 2$ and using $\operatorname{tr}\sigma^0 = -2$, $\operatorname{tr}\sigma^l = 0$, $l = 1, 2, 3$ yields for the Minkowski metric $(-1, 1, 1, 1)$

$$(\Phi, P_0\Phi) > 0$$

i.e. in a supersymmetric theory the energy $H = P_0$ is always positive. This positivity argument does not require any detailed knowledge of the Hilbert space $\mathcal{H}$ which is an imperative of any quantum theory. In this paper we present not only indefinite but also definite (invariant) inner products on $N = 1$ superspace, which are defined on supersymmetric functions on the $N = 1$ superspace, and show that the inherent Hilbert space in supersymmetric theories appears in conjunction with an indefinite (Krein) scalar product. Roughly speaking, each function on superspace can be decomposed in a chiral, antichiral and a transversal contribution. However, it turns out that in order to obtain positivity of the scalar product the transversal contribution has to be subtracted instead of adding it to the chiral/antichiral part.

Despite the previous positivity argument leading to the energy positivity which relies on physical arguments, we prefer for this paper a general mathematical framework and even do not explicitly assume supersymmetry. Comments on physics appear at the end of the paper. We use the notation and conventions of [1] with the only difference that from now on $\sigma^0, \bar{\sigma}^0$ are the identity instead of minus identity (our notation coincides with [2]). In particular our Minkowski metric η^{lm} is $(-1, +1, +1, +1)$. The Fourier transform $\tilde{f}(p)$ of $f(x)$ is defined through

$$f(x) = \frac{1}{(2\pi)^2} \int \mathrm{e}^{\mathrm{i}px} \tilde{f}(p)\,\mathrm{d}p$$

where $px = p_l x^l = p_l \eta^{lm} x_m$.

We use the Weyl spinor formalism in the Van der Waerden notation as in the references cited above although for our purposes 4-component spinors would be better suited (see [3]). Working with Weyl spinors we have to assume for consistency reasons anticommutativity of their components which in our case are regular (test) functions (or distributions). This will be not the case at the point we define sesquilinear form (inner products) by integration on superspace connecting to the usual L^2-scalar product on functions. Certainly this is not a serious problem as it is clear to the reader (see also section 3).

2. The supersymmetric functions

We restrict ourselves to the $N = 1$ superspace. We write the most general superspace (test) function $X = X(z) = X(x, \theta, \bar{\theta})$ as in [1, 2]

$$\begin{aligned} X(z) &= X(x, \theta, \bar{\theta}) \\ &= f(x) + \theta\varphi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) \\ &\quad + \theta\sigma^l\bar{\theta} v_l(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x) \end{aligned} \tag{2.1}$$

where the coefficients are functions of x in Minkowski space of certain regularity which will be specified below (by the end of the paper we will admit distributions too). For the time being suppose that the coefficient functions are in the Schwartz space S of infinitely differentiable (test) functions with faster than polynomial decrease at infinity. For the vector component v we can write equivalently

$$\theta\sigma^l\bar{\theta} v_l = \theta^\alpha\bar{\theta}^{\dot{\alpha}} v_{\alpha\dot{\alpha}}$$

where

$$v_{\alpha\dot{\alpha}} = \sigma^l_{\alpha\dot{\alpha}} v_l,\; v^l = -\tfrac{1}{2}\bar{\sigma}^{l\dot{\alpha}\alpha} v_{\alpha\dot{\alpha}}$$

which is a consequence of the ‘second’ completeness equation

$$\sigma^l_{\alpha\dot{\beta}}\bar{\sigma}_l^{\dot{\gamma}\rho} = -2\delta^\rho_\alpha\delta^{\dot{\gamma}}_{\dot{\beta}}.$$

Let us introduce the supersymmetric covariant (and invariant [1, 2]) derivatives $D, \bar{D}$ with spinorial components $D_\alpha, D^\alpha, \bar{D}_{\dot{\alpha}}, \bar{D}^{\dot{\alpha}}$ given by

$$D_\alpha = \partial_\alpha + \mathrm{i}\sigma^l_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_l \tag{2.2}$$

$$D^\alpha = \epsilon^{\alpha\beta} D_\beta = -\partial^\alpha + \mathrm{i}\sigma^{l\alpha}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_l \tag{2.3}$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^{\alpha}\sigma^{l}_{\alpha\dot{\alpha}}\partial_l \tag{2.4}$$

$$\bar{D}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\beta}} = \bar{\partial}^{\dot{\alpha}} - \mathrm{i}\theta^{\alpha}\sigma^{l\dot{\alpha}}_{\alpha}\partial_l. \tag{2.5}$$

We accept on the way notation like

$$\epsilon^{\alpha\beta}\sigma^{l}_{\beta\dot{\alpha}} = \sigma^{l\alpha}_{\dot{\alpha}}$$

etc but in the end we come back to the canonical index positions $\sigma^l = \left(\sigma^l_{\alpha\dot{\alpha}}\right)$, $\bar{\sigma}^l = (\bar{\sigma}^{l\dot{\alpha}\alpha})$.

Note that D_α does not contain the variable θ and $\bar{D}^{\dot{\alpha}}$ does not contain the variable $\bar{\theta}$ such that we can write at the operatorial level:

$$D^2 = D^{\alpha}D_{\alpha} = -(\partial^{\alpha}\partial_{\alpha} - 2\mathrm{i}\partial_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial^{\alpha} + \bar{\theta}^2\Box) \tag{2.6}$$

$$\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} = -(\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + 2\mathrm{i}\theta^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + \theta^2\Box) \tag{2.7}$$

where

$$\Box = \eta^{lm}\partial_l\partial_m$$

is the d'alembertian, η is the Minkowski metric tensor and

$$\partial_{\alpha\dot{\alpha}} = \sigma^{l}_{\alpha\dot{\alpha}}\partial_l.$$

Here we used the 'first' completeness relation for the Pauli matrices $\sigma, \bar{\sigma}$:

$$\mathrm{Tr}(\sigma^l\bar{\sigma}^m) = \sigma^{l}_{\alpha\dot{\beta}}\bar{\sigma}^{m\dot{\beta}\alpha} = -2\eta^{lm}. \tag{2.8}$$

We make use of the operators [1, 2]

$$c = \bar{D}^2D^2, \qquad a = D^2\bar{D}^2, \qquad T = D^{\alpha}\bar{D}^2D_{\alpha} = \bar{D}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}} = -8\Box + \tfrac{1}{2}(c+a) \tag{2.9}$$

which are used to construct formal projections

$$P_c = \frac{1}{16\Box}c, \qquad P_a = \frac{1}{16\Box}a, \qquad P_T = -\frac{1}{8\Box}T \tag{2.10}$$

on chiral, antichiral and transversal supersymmetric functions. These operators are, at least for the time being, formal because they contain the d'alembertian in the denominator. Problems with the d'alembertian in (2.10) in the denominator will be explained later. Chiral, antichiral and transversal functions are linear subspaces of general supersymmetric functions which are defined by the conditions [1, 2]

$$\bar{D}^{\dot{\alpha}}X = 0, \quad \dot{\alpha} = 1, 2; \qquad D^{\alpha}X = 0, \quad \alpha = 1, 2; \qquad D^2X = \bar{D}^2X = 0$$

respectively. It can be proven that these relations are formally equivalent to the relations

$$P_cX = X, \qquad P_aX = X, \qquad P_TX = X$$

(we mean here that P_i, $i = c, a, T$ are applicable to X and the relations above hold).

We have formally

$$P_i^2 = P_i, \qquad P_iP_j = 0, \qquad i \neq j; \quad i, j = c, a, T$$

and $P_c + P_a + P_T = 1$. Accordingly each supersymmetric function can be formally decomposed into a sum of a chiral, antichiral and transversal contribution (from a rigorous point of view this statement may be wrong and has to be reconsidered because of the problems with the d'alembertian in the denominator; fortunately we will not run into such difficulties as this will be made clear later in the paper).

Let us specify the coefficient functions in (2.1) for the chiral, antichiral and transversal supersymmetric functions.

For the chiral case X_c we have:

$$\begin{aligned} &\bar{\chi} = \psi = n = 0, && v_l = \partial_l(\mathrm{i}f) = \mathrm{i}\partial_l f, \\ &\bar{\lambda} = -\frac{\mathrm{i}}{2}\partial_l\varphi\sigma^l = \frac{\mathrm{i}}{2}\bar{\sigma}^l\partial_l\varphi, && d = \frac{1}{4}\Box f. \end{aligned} \tag{2.11}$$

Here f, φ and m are arbitrary functions. For notation and relations see (2.23)–(2.27).

For the antichiral X_a case:

$$\begin{aligned} &\varphi = \bar\lambda = m = 0, && v_l = \partial_l(-\mathrm{i}f) = -\mathrm{i}\partial_l f, \\ &\psi = \frac{\mathrm{i}}{2}\sigma^l\partial_l\bar\chi = -\frac{\mathrm{i}}{2}\partial_l\bar\chi\bar\sigma^l, && d = \frac{1}{4}\Box f. \end{aligned} \tag{2.12}$$

Here f, $\bar\chi$ and n are arbitrary functions.

For the transversal case X_T [2]:

$$\begin{aligned} &m = n = 0, \qquad \partial_l v^l = 0, && \bar\lambda = \frac{\mathrm{i}}{2}\partial_l\varphi\sigma^l = -\frac{\mathrm{i}}{2}\bar\sigma^l\partial_l\varphi, \\ &\psi = \frac{\mathrm{i}}{2}\partial_l\bar\chi\bar\sigma^l = -\frac{\mathrm{i}}{2}\sigma^l\partial_l\bar\chi, && d = -\frac{1}{4}\Box f. \end{aligned} \tag{2.13}$$

Here $f, \varphi, \bar\chi$ are arbitrary and v satisfies $\partial_l v^l = 0$.

Later on we will need the $\theta^2\bar\theta^2$ coefficients $[\bar X_i X_i](x_1, x_2)$ of the quadratic forms $\bar X_i(x_1, \theta, \bar\theta)X_i(x_2, \theta, \bar\theta)$ for $i = c, a, T$ where $X_0 = X$ is arbitrary supersymmetric. They coincide with the Grassmann integrals

$$\int \mathrm{d}^2\theta_1\,\mathrm{d}^2\bar\theta_1\,\mathrm{d}^2\theta_2\,\mathrm{d}^2\bar\theta_2\,\bar X_i(x_1, \theta_1, \bar\theta_1)\delta^2(\theta_1 - \theta_2)\delta^2(\bar\theta_1 - \bar\theta_2)X_i(x_2, \theta_2, \bar\theta_2) \tag{2.14}$$

where $\delta^2(\theta_1-\theta_2) = (\theta_1-\theta_2)^2$, $\delta^2(\bar\theta_1-\bar\theta_2) = (\bar\theta_1-\bar\theta_2)^2$, $\mathrm{d}^2\theta = \frac{1}{2}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2$, $\mathrm{d}^2\bar\theta = -\frac{1}{2}\,\mathrm{d}\bar\theta^{\bar 1}\mathrm{d}\bar\theta^{\bar 2}$ and are listed below in the order of $i = c, a, T$:

$$\begin{aligned} [\bar X_c X_c](x_1, x_2) = {}& \bar f(x_1)\left(\frac{1}{4}\Box f(x_2)\right) - \bar\varphi(x_1)\left(\frac{\mathrm{i}}{2}\bar\sigma^l\partial_l\varphi(x_2)\right) + \bar m(x_1)m(x_2) \\ & - \frac{1}{2}\partial^l\bar f(x_1)\partial_l f(x_2) - \left(-\frac{\mathrm{i}}{2}\partial_l\bar\varphi(x_1)\bar\sigma^l\right)\varphi(x_2) + \left(\frac{1}{4}\Box\bar f(x_1)\right)f(x_2) \end{aligned} \tag{2.15}$$

$$\begin{aligned} [\bar X_a X_a](x_1, x_2) = {}& \bar f(x_1)\left(\frac{1}{4}\Box f(x_2)\right) - \chi(x_1)\left(\frac{\mathrm{i}}{2}\sigma^l\partial_l\bar\chi(x_2)\right) + \bar n(x_1)n(x_2) \\ & - \frac{1}{2}\partial^l\bar f(x_1)\partial_l f(x_2) - \left(-\frac{\mathrm{i}}{2}\partial_l\chi(x_1)\sigma^l\right)\bar\chi(x_2) + \left(\frac{1}{4}\Box\bar f(x_1)\right)f(x_2) \end{aligned} \tag{2.16}$$

$$\begin{aligned} [\bar X_T X_T](x_1, x_2) = {}& \bar f(x_1)\left(-\frac{1}{4}\Box f(x_2)\right) - \bar\varphi(x_1)\left(-\frac{\mathrm{i}}{2}\bar\sigma^l\partial_l\varphi(x_2)\right) \\ & - \left(\frac{\mathrm{i}}{2}\partial_l\bar\varphi(x_1)\right)\bar\sigma^l\varphi(x_2) - \frac{1}{2}\bar v^l(x_1)v_l(x_2) - \chi(x_1)\left(-\frac{\mathrm{i}}{2}\sigma^l\partial_l\bar\chi(x_2)\right) \\ & - \left(\frac{\mathrm{i}}{2}\partial_l\chi(x_1)\sigma^l\right)\bar\chi(x_2) + \left(-\frac{1}{4}(\Box\bar f(x_1))f(x_2)\right) \end{aligned} \tag{2.17}$$

where we have used relations quoted in (2.23)–(2.27). The conjugate $\bar X$ is given in (2.34).

As a useful exercise let us put $x_1 = x_2$ in $[\bar X_i X_i](x_1, x_2)$, $i = c, a, T$ and compute the integral

$$\int \mathrm{d}^4x[\bar X_i X_i](x).$$

We want to make clear that this computation is done only for pedagogical reasons; we perform it because we will need a similar computation in momentum space (!) at a later stage in this paper. We integrate by parts and use the faster than polynomial decrease of the coefficient functions and of their derivatives to obtain for the chiral case:

$$\int \mathrm{d}^4x[\bar X_c X_c](x) = \int \mathrm{d}^4x\ \bar f(x)\Box f(x) - \int \mathrm{d}^4x\,\bar\varphi(x)\mathrm{i}\bar\sigma^l\partial_l\varphi(x) + \int \mathrm{d}^4x\,\bar m(x)m(x). \tag{2.18}$$

For the antichiral case:

$$\int \mathrm{d}^4x[\bar{X}_a X_a](x) = \int \mathrm{d}^4x\ \bar{f}(x)\Box f(x) - \int \mathrm{d}^4x\ \chi(x)\mathrm{i}\sigma^l\partial_l\bar{\chi}(x) + \int \mathrm{d}^4x\ \bar{n}(x)n(x). \quad (2.19)$$

and for the transversal case:

$$\int \mathrm{d}^4x[\bar{X}_T X_T](x) = -\frac{1}{2}\int \mathrm{d}^4x\ \bar{f}(x)\Box f(x) + \int \mathrm{d}^4x\ \bar{\varphi}(x)\mathrm{i}\bar{\sigma}^l\partial_l\varphi(x)$$
$$+\int \mathrm{d}^4x\ \chi(x)\mathrm{i}\sigma^l\partial_l\bar{\chi}(x) - \frac{1}{2}\int \mathrm{d}^4x\ \bar{v}^l(x)v_l(x). \quad (2.20)$$

Certainly the best we can expect in our paper is to find a Hilbert space structure on supersymmetric functions such that the decomposition formally suggested by $P_c + P_a + P_T = 1$ is a direct orthogonal sum of chiral, antichiral and transversal functions, but this is definitely not the case as will be clear soon. In this paper we are going to uncover the exact mathematical structure of this decomposition in its several variants. This will be done by explicit computations. We start computing the action of the operators D_α, D^α, $\bar{D}_{\dot{\alpha}}$, $\bar{D}^{\dot{\alpha}}$, D^2, $\bar{D}^2$, c, a, T on X. Usually in physics one does not need the results of all these elementary but long computations in an explicit way and this is the reason they are not fully recorded in the literature. It turns out that for our purposes we need at least some of them.

For a given X as in (2.1) the expressions $D_\beta X$, $D^\gamma X$, $\bar{D}^{\dot{\beta}} X$, $\bar{D}^{\dot{\gamma}} X$ are easily computed but are not given explicitly here because they are in fact not necessary in order to compute higher covariant derivatives used in this paper (in order to compute higher derivatives we use (2.6) and (2.7)).

We start by recording the results for D^2, $\bar{D}^2$ applied on X:

$$\bar{D}^2X = -4n + \theta(-4\psi - 2\mathrm{i}\sigma^l\partial_l\bar{\chi}) + \theta^2(-4d - 2\mathrm{i}\partial_l v^l - \Box f) + \theta\sigma^l\bar{\theta}(-4\mathrm{i}\partial_l n)$$
$$+\theta^2\bar{\theta}(-2\mathrm{i}\bar{\sigma}^l\partial_l\psi - \Box\bar{\chi}) + \theta^2\bar{\theta}^2(-\Box n)$$
$$D^2X = -4m + \bar{\theta}(-4\bar{\lambda} - 2\mathrm{i}\bar{\sigma}^l\partial_l\varphi) + \bar{\theta}^2(-4d + 2\mathrm{i}\partial_l v^l - \Box f) + \theta\sigma^l\bar{\theta}(4\mathrm{i}\partial_l m)$$
$$+\bar{\theta}^2\theta(-2\mathrm{i}\sigma^l\partial_l\bar{\lambda} - \Box\varphi) + \theta^2\bar{\theta}^2(-\Box m)$$

or in a more suggestive way taking into account the chirality/antichirality of $\bar{D}^2X$, D^2X (see (2.11), (2.12)):

$$\bar{D}^2X = -4n + \theta(-4\psi - 2\mathrm{i}\sigma^l\partial_l\bar{\chi}) + \theta^2(-4d - 2\mathrm{i}\partial_l v^l - \Box f) + \theta\sigma^l\bar{\theta}(-4\mathrm{i}\partial_l n)$$
$$+\theta^2\bar{\theta}\left(\tfrac{1}{2}\mathrm{i}\bar{\sigma}^l\partial_l\right)(-4\psi - 2\mathrm{i}\sigma^l\partial_l\bar{\chi}) + \theta^2\bar{\theta}^2(-\Box n) \quad (2.21)$$
$$D^2X = -4m + \bar{\theta}(-4\bar{\lambda} - 2\mathrm{i}\bar{\sigma}^l\partial_l\varphi) + \bar{\theta}^2(-4d + 2\mathrm{i}\partial_l v^l - \Box f) + \theta\sigma^l\bar{\theta}(4\mathrm{i}\partial_l m)$$
$$+\bar{\theta}^2\theta\left(\tfrac{1}{2}\mathrm{i}\sigma^l\partial_l\right)(-4\bar{\lambda} - 2\mathrm{i}\bar{\sigma}^l\partial_l\varphi) + \theta^2\bar{\theta}^2(-\Box m). \quad (2.22)$$

We have used the following notation and relations (see for instance the standard references mentioned above):

$$(\psi\sigma^l)_{\dot{\beta}} = \psi^\alpha\sigma^l_{\alpha\dot{\beta}}, \qquad (\sigma^l\bar{\chi})_\beta = \sigma^l_{\beta\dot{\rho}}\bar{\chi}^{\dot{\rho}}, \qquad (\bar{\chi}\bar{\sigma}^l)^\alpha = \bar{\chi}_{\dot{\rho}}\bar{\sigma}^{l\dot{\rho}\alpha}, \qquad (\bar{\sigma}^l\psi)^{\dot{\alpha}} = \bar{\sigma}^{l\dot{\alpha}\beta}\psi_\beta \quad (2.23)$$

with $(\sigma^l\bar{\chi})^\alpha = -(\bar{\chi}\bar{\sigma}^l)^\alpha$ etc as well as

$$\psi\sigma^l\bar{\chi} = \psi^\alpha\sigma^l_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}} = -\bar{\chi}\bar{\sigma}^l\psi = -\bar{\chi}_{\dot{\alpha}}\bar{\sigma}^{l\dot{\alpha}\beta}\psi_\beta \quad (2.24)$$

where $\bar{\sigma}^l_{\dot{\alpha}\beta} = \sigma^l_{\beta\dot{\alpha}}$.

As far as the complex conjugation is concerned we have:

$$(\psi\sigma^l)^*_{\dot{\alpha}} = (\sigma^l\bar{\psi})_\alpha, \qquad (\bar{\chi}\bar{\sigma}^l)^{\alpha *} = (\bar{\sigma}\chi)^{\dot{\alpha}}, \qquad (\psi\sigma^l\bar{\chi})^* = \chi\sigma^l\bar{\psi} \quad (2.25)$$

where $*$ is the complex conjugation defined such that

$$(\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}} \tag{2.26}$$

$$(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}. \tag{2.27}$$

The unusual properties of the Grassmann derivative were taken into account; in particular $\partial_\alpha^* = -\bar{\partial}_{\dot{\alpha}}$ etc.

As expected $\bar{D}^2 X$ and $D^2 X$ are chiral and antichiral functions respectively. We continue with $c = \bar{D}^2 D^2$, $a = D^2 \bar{D}^2$:

$$\begin{aligned} cX = \bar{D}^2 D^2 X &= 16d - 8\mathrm{i}\partial_l v^l + 4\Box f + \theta(8\Box\varphi + 16\mathrm{i}\sigma^l\partial_l\bar{\lambda}) \\ &\quad + \theta^2(16\Box m) + \theta\sigma^l\bar{\theta}(16\mathrm{i}\partial_l d + 8\partial_l\partial_m v^m + 4\mathrm{i}\partial_l\Box f) \\ &\quad + \theta^2\bar{\theta}(8\Box\bar{\lambda} + 4\mathrm{i}\bar{\sigma}^l\partial_l\Box\varphi) + \theta^2\bar{\theta}^2(4\Box d - 2\mathrm{i}\partial_l\Box v^l + \Box^2 f) \end{aligned} \tag{2.28}$$

$$\begin{aligned} aX = D^2\bar{D}^2 X &= 16d + 8\mathrm{i}\partial_l v^l + 4\Box f + \bar{\theta}(8\Box\bar{\chi} + 16\mathrm{i}\bar{\sigma}^l\partial_l\psi) \\ &\quad + \bar{\theta}^2(16\Box n) + \theta\sigma^l\bar{\theta}(-16\mathrm{i}\partial_l d + 8\partial_l\partial_m v^m - 4\mathrm{i}\partial_l\Box f) \\ &\quad + \bar{\theta}^2\theta(8\Box\psi + 4\mathrm{i}\sigma^l\partial_l\Box\bar{\chi}) + \theta^2\bar{\theta}^2(4\Box d + 2\mathrm{i}\partial_l\Box v^l + \Box^2 f) \end{aligned} \tag{2.29}$$

and finally obtain $T = -8\Box + \frac{1}{2}(c + a)$ applied on X as follows:

$$\begin{aligned} TX &= 16d - 4\Box f + \theta(-4\Box\varphi + 8\mathrm{i}\sigma^l\partial_l\bar{\lambda}) + \bar{\theta}(-4\Box\bar{\chi} + 8\mathrm{i}\bar{\sigma}^l\partial_l\psi) \\ &\quad + \theta\sigma^l\bar{\theta}(8\partial_l\partial^m v_m - 8\Box v_l) + \theta^2\bar{\theta}(-4\Box\bar{\lambda} + 2\mathrm{i}\bar{\sigma}^l\partial_l\Box\varphi) \\ &\quad + \bar{\theta}^2\theta(-4\Box\psi + 2\mathrm{i}\sigma^l\partial_l\Box\bar{\chi}) + \theta^2\bar{\theta}^2(-4\Box d + \Box^2 f) \end{aligned} \tag{2.30}$$

or

$$\begin{aligned} TX &= 16d - 4\Box f + \theta(-4\Box\varphi + 8\mathrm{i}\sigma^l\partial_l\bar{\lambda}) + \bar{\theta}(-4\Box\bar{\chi} + 8\mathrm{i}\bar{\sigma}^l\partial_l\psi) \\ &\quad + \theta\sigma^l\bar{\theta}(8\partial_l\partial^m v_m - 8\Box v_l) + \theta^2\bar{\theta}\left(-\frac{\mathrm{i}}{2}\bar{\sigma}^l\partial_l\right)(-4\Box\varphi + 8\mathrm{i}\sigma^l\partial_l\bar{\lambda}) \\ &\quad + \bar{\theta}^2\theta\left(-\frac{\mathrm{i}}{2}\sigma^l\partial_l\right)(-4\Box\bar{\chi} + 8\mathrm{i}\bar{\sigma}^l\partial_l\psi) + \theta^2\bar{\theta}^2(-4\Box d + \Box^2 f). \end{aligned} \tag{2.31}$$

Here we have used the relations

$$(\sigma\partial)(\bar{\sigma}\partial) = (\bar{\sigma}\partial)(\sigma\partial) = -\Box 1_{2\times 2} \tag{2.32}$$

where we briefly write

$$\sigma\partial = \sigma^l\partial_l, \qquad \bar{\sigma}\partial = \bar{\sigma}^l\partial_l. \tag{2.33}$$

Relation (2.32) as well as relation (2.8) follows from

$$\sigma^l\bar{\sigma}^m + \sigma^m\bar{\sigma}^l = -2\eta^{lm}1_{2\times 2}.$$

where $1_{2\times 2}$ is the unit 2×2 matrix. Written in the spinor notation it reads

$$\sigma^l_{\alpha\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\beta} + \sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{l\dot{\alpha}\beta} = -2\eta^{lm}\delta^\beta_\alpha.$$

As expected $\bar{D}^2 D^2 X$ is chiral, $D^2\bar{D}^2 X$ is antichiral and TX is transversal. The transversality (2.13) of TX was put in evidence in (2.31).

In order to construct inner products in integral form we also need the conjugates $\bar{X}, \overline{\bar{D}^2 X}, \overline{D^2 X}$, etc of $X, \bar{D}^2 X, D^2 X$ etc where the conjugation includes besides the usual complex conjugation the Grassman conjugation too. We have

$$\begin{aligned} \bar{X} &= \bar{X}(x,\theta,\bar{\theta}) \\ &= \bar{f}(x) + \theta\chi(x) + \bar{\theta}\bar{\varphi}(x) + \theta^2\bar{n}(x) + \bar{\theta}^2\bar{m}(x) \\ &\quad + \theta\sigma^l\bar{\theta}\bar{v}_l(x) + \theta^2\bar{\theta}\bar{\psi}(x) + \bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}^2\bar{d}(x) \end{aligned} \tag{2.34}$$

where $\bar{f}$, χ, $\bar{\varphi}$, etc are the complex conjugate functions to f, $\bar{\chi}$, φ, etc. Note that if X is chiral then $\bar{X}$ is antichiral and vice versa. If X is transversal then $\bar{X}$ is transversal. Although not absolutely necessary we record here other expressions too which can be used to give alternative proofs of results to follow by making use of partial integration in superspace. They are (use $(\chi\sigma^l\bar{\psi})^* = \psi\sigma^l\bar{\chi}$ where $*$ is complex conjugation, which could also have been written as a bar):

$$\overline{\bar{D}^2X} = D^2\bar{X} = -4\bar{n} + \bar{\theta}(-4\bar{\psi} - 2\mathrm{i}\bar{\sigma}^l\partial_l\chi) + \bar{\theta}^2(-4\bar{d} + 2\mathrm{i}\partial_l\bar{v}^l - \Box\bar{f}) \\ + \theta\sigma^l\bar{\theta}(4\mathrm{i}\partial_l\bar{n}) + \bar{\theta}^2\theta(-2\mathrm{i}\sigma^l\partial_l\bar{\psi} - \Box\chi) + \theta^2\bar{\theta}^2(-\Box\bar{n}) \tag{2.35}$$

$$\overline{D^2X} = \bar{D}^2\bar{X} = -4\bar{m} + \theta(-4\lambda - 2\mathrm{i}\sigma^l\partial_l\bar{\varphi}) + \theta^2(-4\bar{d} - 2\mathrm{i}\partial_l\bar{v}^l - \Box\bar{f}) \\ + \theta\sigma^l\bar{\theta}(-4\mathrm{i}\partial_l\bar{m}) + \theta^2\bar{\theta}(-2\mathrm{i}\bar{\sigma}^l\partial_l\lambda - \Box\bar{\varphi}) + \theta^2\bar{\theta}^2(-\Box\bar{m}) \tag{2.36}$$

or in a more suggestive way as chiral and antichiral functions respectively

$$\overline{\bar{D}^2X} = -4\bar{n} + \bar{\theta}\bar{\eta} + \bar{\theta}^2(-4\bar{d} + 2\mathrm{i}\partial_l\bar{v}^l - \Box\bar{f}) + \theta\sigma^l\bar{\theta}(4\mathrm{i}\partial_l\bar{n}) + \theta^2\bar{\theta}\left(\frac{\mathrm{i}}{2}\sigma^l\partial_l\right)\bar{\eta} + \theta^2\bar{\theta}^2(-\Box\bar{n}) \tag{2.37}$$

$$\overline{D^2X} = -4\bar{m} + \theta\xi + \theta^2(-4\bar{d} - 2\mathrm{i}\partial_l\bar{v}^l - \Box\bar{f}) + \theta\sigma^l\bar{\theta}(4\mathrm{i}\partial_l\bar{m}) + \theta^2\bar{\theta}\left(\frac{\mathrm{i}}{2}\bar{\sigma}^l\partial_l\right)\xi + \theta^2\bar{\theta}^2(-\Box\bar{m}) \tag{2.38}$$

where in (2.37) and (2.38) we have set

$$\xi = -4\lambda - 2\mathrm{i}\sigma^l\partial_l\bar{\varphi}, \qquad \bar{\eta} = -4\bar{\psi} - 2\mathrm{i}\bar{\sigma}^l\partial_l\chi.$$

Further

$$\overline{cX} = \overline{\bar{D}^2D^2X} = \bar{D}^2D^2\bar{X} \tag{2.39}$$

$$\overline{aX} = \overline{D^2\bar{D}^2X} = D^2\bar{D}^2\bar{X} \tag{2.40}$$

and finally

$$\overline{TX} = \bar{T}\bar{X} = T\bar{X} \tag{2.41}$$

or

$$\overline{TX} = 16\bar{d} - 4\Box\bar{f} + \theta\xi + \bar{\theta}\bar{\eta} + \theta\sigma^l\bar{\theta}(8\partial_l\partial^m\bar{v}_m - 8\Box\bar{v}_l) + \theta^2\bar{\theta}\left(-\frac{\mathrm{i}}{2}\bar{\sigma}^l\partial_l\right)\xi \\ + \bar{\theta}^2\theta\left(-\frac{\mathrm{i}}{2}\sigma^l\partial_l\right)\bar{\eta} + \theta^2\bar{\theta}^2(-4\Box\bar{d} + \Box^2\bar{f}) \tag{2.42}$$

where in (2.42):

$$\xi = -4\Box\chi + 8\mathrm{i}\sigma^l\partial_l\bar{\psi}, \qquad \bar{\eta} = -4\Box\bar{\varphi} + 8\mathrm{i}\bar{\sigma}^l\partial_l\lambda.$$

We start to look for (invariant) supersymmetric kernel functions $K(z_1, z_2) = K(x_1, \theta_1, \bar{\theta}_1; x_2, \theta_2, \bar{\theta}_2)$ which formally induce inner products on supersymmetric functions by

$$(X_1, X_2) = \int \mathrm{d}^8z_1\,\mathrm{d}^8z_2\bar{X}_1(z_1)K(z_1, z_2)X_2(z_2) = \int \bar{X}_1KX_2 \tag{2.43}$$

where the bar on the rhs means conjugation (including Grassmann), $z_i = (x_i, \theta_i, \bar{\theta}_i)$ and $\mathrm{d}^8z = \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}$. On the rhs of the last equality we have used a sloppy but concise notation of the integral under study. The simplest choice for K would be the identity kernel

$K(z_1, z_2) = k(z_1 - z_2) = \delta^2(\theta_1 - \theta_2)\delta^2(\bar\theta_1 - \bar\theta_2)\delta^4(x_1 - x_2)$ but it turns out that this choice is not sound. We settle soon for more appropriate choices. Formally we have if $\bar K = K$:

$$\overline{(X_1, X_2)} = (\bar X_2, \bar X_1)$$

where the bars include Grassmann conjugation. The action of K on X is defined formally as

$$Y_K(z_1) = (KX)(z_1) = \int \mathrm{d}^8 z_2\, K(z_1, z_2) X(z_2).$$

Note that the general dependence of K on z_1, z_2 we admit is not necessarily through the difference $z_1 - z_2$. We assume that the coefficient functions of the supersymmetric functions involved belong to the Schwartz function space S of infinitely differentiable rapidly decreasing functions.

Now we are starting to induce positivity of the inner product by a proper choice of the kernel K. By positivity in this section we mean non-negativity. The first candidate is

$$K(z_1, z_2) = K_0(z_1 - z_2) = \delta^2(\theta_1 - \theta_2)\delta^2(\bar\theta_1 - \bar\theta_2) D^+(x_1 - x_2) \tag{2.44}$$

where $\delta^2(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2$, $\delta^2(\bar\theta_1 - \bar\theta_2) = (\bar\theta_1 - \bar\theta_2)^2$ are the supersymmetric δ-functions and $D^+(x)$ is the Fourier transform of a positive Lorentz invariant measure $\mathrm{d}\rho(p)$ supported in the backward light cone $\bar V^-$:

$$D^+(x) = \frac{1}{(2\pi)^2} \int \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}\rho(p) \tag{2.45}$$

which is of polynomial growth, i.e. there is an integer n such that

$$\int \frac{\mathrm{d}\rho(p)}{(1 + |p|^2)^n} < \infty \tag{2.46}$$

where $|p| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}$. Here px is the 'most positive' Minkowski scalar product. Usually (for instance in quantum field theory) the Minkowski scalar product is 'most negative' and as a consequence the measure $\mathrm{d}\rho(p)$ is concentrated in the forward light cone $\bar V^+$. The special kernel K_0 depends only on the difference $z_1 - z_2$. In order to understand the idea behind this choice note first that for f and g functions of x in S the integral

$$(f, g) = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar f(x) D^+(x - y) g(y) \tag{2.47}$$

where $D^+(x)$ is given by (2.45) induces a positive definite scalar product (certainly in order to exclude zero vectors we have to require the support of f and g in momentum space in $\bar V^-$ to be concentrated on the support of $\mathrm{d}\rho(p)$ which is equivalent to factoring out zero vectors and completion in (2.47)). Indeed the right-hand side of (2.47) equals in momentum space $\int \bar{\tilde f}(p)\tilde g(p)\, \mathrm{d}\rho(p)$ where $\tilde f$ is the Fourier transform of f given by $f(x) = \frac{1}{(2\pi)^2}\int \mathrm{e}^{\mathrm{i}px} \tilde f(p)\, \mathrm{d}p$. Note further that positivity is preserved if we multiply the measure $\mathrm{d}\rho(p)$ by $-p^2$ or for the case of two-spinor functions f and g by σp or $\bar\sigma p$. In configuration space it means that we can accommodate the operators $\Box$ and $-\mathrm{i}\sigma\partial$, $-\mathrm{i}\bar\sigma\partial$ in the kernel of the integral without spelling out the positivity (we have as usually $\frac{1}{\mathrm{i}}\partial = p$).

It is clear that in spite of the positivity properties induced by the kernel D^+ the scalar product in (2.43) with kernel (2.44) cannot be positive definite in superspace for general coefficient functions (for $X_1 = X_2 = X$) because the coefficient functions are mixed up in the process of Grassmann integration in an uncontrolled way. Fortunately there are other kernels deduced from K_0 which do the job. In order to keep the technicalities aside for the moment let us assume that the measure $\mathrm{d}\rho(p)$, besides being of polynomial growth, satisfies the stronger condition

$$\left| \int \frac{1}{p^2} \frac{\mathrm{d}\rho(p)}{(1 + |p|^2)^n} \right| < \infty \tag{2.48}$$

with the integer n appearing in (2.46).

Certainly the condition above is relatively strong; it allows measures like $\mathrm{d}\rho(p) = \theta(-p_0)\delta(p^2 + m^2)\,\mathrm{d}p$ with $m > 0$ but excludes the cases $m = 0$ (in physics the massive and massless cases respectively). The case $m = 0$ will be studied at the end of this section.

We arrived at the level of explaining our message. For this we introduce besides $K_0(z_1 - z_2)$ three other kernels as follows,

$$K_c(z_1, z_2) = P_c K_0(z_1 - z_2) \tag{2.49}$$

$$K_a(z_1, z_2) = P_a K_0(z_1 - z_2) \tag{2.50}$$

$$K_T(z_1, z_2) = -P_T K_0(z_1 - z_2) \tag{2.51}$$

with actions

$$Y_i(z_1) = (K_i X)(z_1) = \int \mathrm{d}^8 z_2\, K_i(z_1, z_2) X(z_2).$$

In (2.49)–(2.51) the operators P_i are understood to act on the first variable z_1 (see also (2.52)–(2.57) to follow). Condition (2.48) makes the formal definition $Y = \int K X$ (with K replaced by one of the derived kernels K_i, $i = c, a, T$ as written above) safe from a rigorous point of view because it takes care of the d'alembertian in the denominators introduced by the formal projections P_i, $i = c, a, T$. We will remove this condition soon by slightly restricting the set of supersymmetric (test) functions but let us keep it for the time being. Note that the projections destroy the translation invariance in the Grassmann variables but not in the space coordinates. Because $P_i, i = c, a, T$ contain Grassmann variables and derivatives thereof we have to specify on which variables they act in $K_0(z_1 - z_2)$. By convention let us define by $D_1^2 K_0(z_1 - z_2)$, $\bar{D}_1^2 K_0(z_1 - z_2)$, $T_1 K_0(z_1 - z_2)$ the action of the operators D^2 and $\bar{D}^2, T$ on $K_0(z_1 - z_2)$ on the first variable respectively and by $D_2^2 K_0(z_1 - z_2)$, $\bar{D}_2^2 K_0(z_1 - z_2)$, $T_2 K_0(z_1 - z_2)$ the action of these operators on the second variable. If indices are not specified, we understand the action on the first variable.

It can be proven (for similar computations see for instance [2]) that

$$D_1^2 K_0(z_1 - z_2) = D_2^2 K_0(z_1 - z_2) \tag{2.52}$$

$$\bar{D}_1^2 K_0(z_1 - z_2) = \bar{D}_2^2 K_0(z_1 - z_2) \tag{2.53}$$

$$D_1^2 \bar{D}_1^2 K_0(z_1 - z_2) = \bar{D}_2^2 D_2^2 K_0(z_1 - z_2) \tag{2.54}$$

$$\bar{D}_1^2 D_1^2 K_0(z_1 - z_2) = D_2^2 D_2^2 K_0(z_1 - z_2) \tag{2.55}$$

$$T_1 K_0(z_1 - z_2) = T_2 K_0(z_1 - z_2) \tag{2.56}$$

$$\bar{T}_1 K_0(z_1 - z_2) = \bar{T}_2 K_0(z_1 - z_2) \tag{2.57}$$

where in fact the relations (2.56), (2.57) coincide because $\bar{T} = T$. We have used

$$\left[D_1^2, D_2^2\right] = 0, \qquad \left[\bar{D}_1^2, \bar{D}_2^2\right] = 0.$$

Note the minus sign in front of P_T in (2.51) which will be of utmost importance for us. Because of it the kernels $K_i, i = c, a, T$ do not sum up to K. This is at the heart of the matter, while at the same time not too embarrassing. We will prove by direct computation that the kernels $K_i, i = c, a, T$ produce, each for itself, a positive definite scalar product in the space of supersymmetric functions (at this stage we prove only nonnegativity; the problem of zero vectors is pushed to section 3). Whereas this assertion is to be expected for K_i for $i = c, a$, the minus sign in K_T comes as a surprise. It will be the reason for the natural Krein (more precisely Hilbert–Krein) structure of the $N = 1$ supersymmetry which we are going to uncover (first under the restrictive condition (2.48) on the measure). Denoting by $(.,.)_i, i = 0, c, a, T$ the inner products induced by the kernels $K_i, i = 0, c, a, T$:

$$(X_1, X_2)_i = \int \bar{X}_1 K_i X_2 \tag{2.58}$$

we could compute them by brute force using expressions (2.28)–(2.30) but it is not easy to get the positive definiteness of these inner products in the cases $i = c, a, T$. Alternatively we will proceed as follows. Let us start with the cases $i = c, a$. We use (2.52)–(2.57) and integrate by parts in superspace (see for instance [2]). This gives (in the sloppy integral notation) by partial integration in superspace

$$(X_1, X_2)_c = \int \bar{X}_1 K_c X_2 = \int \bar{X}_1 P_c K_0 X_2 = \left(D_1^2 X_1, \frac{1}{16\Box} D_2^2 X_2 \right)_0 \tag{2.59}$$

$$(X_1, X_2)_a = \int \bar{X}_1 K_a X_2 = \int \bar{X}_1 P_a K_0 X_2 = \left(\bar{D}_1^2 X_1, \frac{1}{16\Box} \bar{D}_2^2 X_2 \right)_0 \tag{2.60}$$

where we have also used $\overline{D^2 X} = \bar{D}^2 \bar{X}$, etc. The last equality follows from obvious ones supplemented by $\overline{\mathrm{i}\sigma^l \partial_l \bar{\varphi}} = (\mathrm{i}\sigma^l \partial_l \bar{\varphi})^* = -\mathrm{i}\partial_l \varphi \sigma^l = \mathrm{i}\bar{\sigma}^l \partial_l \varphi$, etc. In (2.59) $\bar{D}^2$ from P_c was moved to $\bar{X}_1$, the remaining D^2 (acting on K_0 on the first variable) was transferred by (2.52) to the second variable on K_0, and then moved on X_2 such that finally we get the last expression. The same procedure was applied for (2.60). The d'alembertian in the denominator can be absorbed in Fourier space in the measure $\mathrm{d}\rho(p)$ which is supposed to satisfy condition (2.48). Using the δ-function property in the Grassmann variables in K_0 we see that for instance in the antichiral case we get for $X_1 = X_2 = X$

$$\begin{aligned}(X, X)_a &= \int \bar{X} K_a X \\ &= \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2 [\overline{(\bar{D}^2 X)}(\bar{D}^2 X)](x_1, x_2) \frac{1}{16\Box} D^+(x_1 - x_2)\end{aligned} \tag{2.61}$$

where [.], as before, gives the coefficient of the highest power in the Grassmann variables.

Note that $\bar{D}^2 X$ is chiral such that for $[\overline{(\bar{D}^2 X)}(\bar{D}^2 X)](x_1, x_2)$ we can apply (2.15). We integrate by parts in the usual coordinates using the faster than polynomial decrease of the involved functions and their derivatives and obtain in momentum space

$$\int \bar{X} K_a X = \int [\overline{\tilde{f}_c}(p) \tilde{f}_c(p) + \overline{\tilde{\varphi}_c}(p)(\sigma p)\tilde{\varphi}_c + \overline{\tilde{m}_c}(p)\tilde{m}_c(p))] \left(\frac{1}{-p^2} \right) \mathrm{d}\rho(p) \tag{2.62}$$

where f_c, φ_c, m_c are the coefficients of the chiral $\bar{D}^2 X$ given by (2.21). We have used the translation invariance of $D^+(x)$ which enables us to read up the result in momentum space from the computation conducting to (2.18) which was performed in coordinate space (this is an unusual way to keep track of the δ-function in momentum space generated by translation invariance which quickly gives the result).

From (2.62) we obtain by inspection the positivity of $\int \bar{X} K_a X = (X, P_a X)_0$. We use the positivity of $-p^2, \sigma p$ and $\bar{\sigma} p$. The same argument works for the chiral integral $\int \bar{X} K_c X = (X, P_c X)_0$.

Now we go over to the transversal integral $\int \bar{X}_1 K_T X_2$. Here we cannot split the kernel in a useful way as we did in the chiral and antichiral cases but the following similar procedure can be applied.

We write using $P_T^2 = P_T$, relation (2.56) and integration by parts in superspace

$$-(X_1, X_2)_T = -\int \bar{X}_1 K_T X_2 = \int \bar{X}_1 P_T K_0 X_2 = \int \bar{X}_1 P_T^2 K_0 X_2$$
$$= (P_T X_1, P_T X_2)_0 = \frac{1}{64}\left(\frac{1}{\Box} T X_1, \frac{1}{\Box} T X_2\right)_0. \tag{2.63}$$

Here, as in the antichiral case above, one of P_T in P_T^2 acting on the first variable was moved to $\bar{X}_1$ and the second one was pushed through K_0 (modulo changing the variable) to X_2. In (2.63) we take $X_1 = X_2 = X$, integrate the $\theta, \bar{\theta}$-variables and use for $[(\overline{TX})(TX))(x_1, x_2)$ expression (2.17). We can now use (2.20) by analogy in momentum space as above. Note that by integration by parts we have enough derivatives in the numerator in order to cancel one of the two inverse d'alembertians in (2.63). By (2.48) the second d'alembertian is under control and the computation is safe. We propose to the reader to go this way in order to explicitly convince himself that the integral $-\int \bar{X}_1 K_T X_2 = \int \bar{X}_1 P_T K_0 X_2$ (in contradistinction to the chiral/antichiral case) is negative for $X_1 = X_2$! A hint is necessary. Indeed the only contribution which has to be looked up beyond the chiral/antichiral case is the vector contribution stemming from v-coefficients of the transversal supersymmetric function and this produces a negative contribution. In fact the negativity of the transversal contribution rests on the following property in momentum space. Let $v(p) = (v_l(p))$ be a vector function (not necessary real) such that $p_l v^l(p) = 0$. It means that the vector with components $v_l(p)$ is orthogonal (in the Euclidean meaning) to the (real) vector p_l. But the momentum vector p is confined to the light cone (it must be in the support of $\mathrm{d}\rho(p)$) such that the vector function $v(p)$ must satisfy $\bar{v}^l(p)v_l(p) \geqslant 0$. Moreover if $\mathrm{d}\rho$ intersects the light cone $p^2 = 0$ the equality may be realized. We repeat here an old argument which was recognized in the frame of the rigorous version of the Gupta–Bleuler quantization in physics [4, 5].

The last part of this section is dedicated to the more delicate question of abolishing the unpleasant restrictive condition (2.48) such that we can include in our analysis, from a physical point of view, the interesting 'massless' case. From the consideration above it is clear that this is generally not possible. More precisely, if we want to retain the interpretation of supersymmetric quantum fields as operator-valued (super)distributions, as this is the case for the usual quantum fields [6] (an interpretation which we subscribe to), we are forced to restrict the set of allowed test functions such that the d'alembertian in the denominator is annihilated. Restricting the set of test functions in quantum field theory is not a problem and is not at all new; it appeared a long time ago in the rigorous discussion of the Gupta–Bleuler quantization [4, 5]. In order to motivate the restriction of super-test functions to follow let us show, following [5], that the classical Gupta–Bleuler quantization as presented in the physical literature is equivalent to a restriction of the set of usual test functions. Indeed the main point of the Gupta–Bleuler method is the 'subsidiary condition'

$$\partial^\mu v_\mu^{(-)} \Phi = 0$$

on the annihilation part of $\partial^\mu v_\mu$ where $v(x) = v = (v_\mu)$ is the massless vector field and $\Phi = |\Phi\rangle$ are the states selected by the subsidiary condition to be the physical ones. From the physical point of view this condition eliminates the scalar and the transversal 'photons' which tend to drive the metric into an indefinite one. Smearing with the test function $f(x) = f = (f^\mu)$ the vector field can be written as

$$v(f) = \int v_\mu(x) f^\mu(x)\,\mathrm{d}x$$

whereas the annihilation part of the vector field has the following Fock space representation in momentum space,

$$(v^{(-)}(f)\Phi)^{(n)}_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n) = \int \frac{\mathrm{d}^3k}{k^0}\tilde{f}^{\mu}(k)\Phi^{(n+1)}_{\mu,\mu_1,\ldots,\mu_n}(k,k_1,\ldots,k_n).$$

It is clear now that the (mathematical) restrictive divergence condition $\operatorname{div} f = 0$ on the test functions f on which the massless vector field v is defined is equivalent to the physical subsidiary condition above. This follows from the basic rule of distribution theory which enables us to transfer a differential operator (in this case the divergence) from a distribution to the test function on which it is applied. Certainly the creation part of the vector field induces states in the Fock space satisfying the divergence condition too.

Before going over to the supersymmetric case let us remark that besides manifest Lorentz invariance the Gupta–Bleuler quantization of the vector field in the variant above hides a gauge fixing which is exactly the Feynman gauge known from the path integral formalism with a Stueckelberg Lagrangian. We can induce other gauges too (such as the Landau or the unitary gauge) inside a (gentle) family of gauges by substituting $\tilde{f}^{\mu}(k)$ on the rhs of the Fock space representation of $v^{(-)}(f)$ by $\tilde{f}^{\mu}(k) - (1-\alpha)k^{\mu}k_{\nu}\tilde{f}^{\nu}(k)$ with an arbitrary constant α. The Feyman gauge corresponds to $\alpha = 1$. Having defined a family of gauges one can discuss the problem of the gauge invariance inside this family [5].

Now we come to the supersymmetric case. Suppose that the coefficient functions in (2.1) satisfy the following restrictive conditions:

$$d(x) = \Box D(x) \tag{2.64}$$

$$\bar{\lambda}(x) = \mathrm{i}\bar{\sigma}^l\partial_l\Lambda(x) \tag{2.65}$$

$$\psi(x) = \mathrm{i}\sigma^l\partial_l\bar{\psi}(x) \tag{2.66}$$

$$v(x) = \operatorname{grad}\rho(x) + \omega(x), \qquad \operatorname{div}\omega(x) = 0 \tag{2.67}$$

where $D(x), \Lambda(x), \Psi(x), \rho(x), \omega(x)$ are arbitrary functions (in S). In the last equation $\operatorname{grad}\rho = (\partial_l\rho)$, $\operatorname{div}\omega = \partial_l\omega^l$.

The functions $\rho(x), \omega(x)$ can be constructed as follows: let ρ be a solution of $\Box\rho = \operatorname{div} v$ and let $\omega = v - \operatorname{grad}\rho$. Then $v = \operatorname{grad}\rho + \omega$ with $\operatorname{div}\omega = \operatorname{div} v - \Box\rho = 0$. The divergence condition $\operatorname{div} v = 0$ is equivalent to the wave equation $\Box\rho = 0$.

We claim that under these conditions the results above concerning the positivity in the chiral/antichiral sectors and negativity in the transversal sector remain valid without the restrictive condition (2.48) on the measure $\mathrm{d}\rho$. Conditions (2.64)–(2.67) produce the missing d'alembertian in

$$\int \bar{X}_1K_iX_2, i = c, a, T \tag{2.68}$$

such that condition (2.48) becomes superfluous. Indeed let us consider for example the chiral case (with the antichiral kernel K_a). From (2.21) we see that the following expressions appear in the integral (2.62):

$$(-4\bar{n})\Box(-4n)$$
$$(-4\bar{\psi} - 2\mathrm{i}\bar{\sigma}^l\partial_l\chi)(\mathrm{i}\bar{\sigma}^n\partial_n)(-4\psi - 2\mathrm{i}\sigma^m\partial_m\bar{\chi})$$
$$(-4\bar{d} + 2\mathrm{i}\partial_l\bar{v}^l - \Box\bar{f})(-4d - 2\mathrm{i}\partial_m v^m - \Box f).$$

It is clear that under conditions (2.64)–(2.67) the missing d'alembertian in integral (2.60) can be factorized such that condition (2.48) on the measure $\mathrm{d}\rho$ is no longer needed. The result remains positive. Similar arguments work for the chiral and transversal cases. In the transversal case the interference between ρ and ω in $\bar{v}^l v_l$ disappears (because div $\omega = 0$) and one can use (besides the positivity of the d'alembertian) again the Gupta–Bleuler argument with div $\omega = 0$.

An interesting point is to search for a physical interpretation of the restriction conditions (2.64)–(2.67) on the (super) test functions. First let us remark that it is easy to motivate these conditions from a technical point of view. Indeed considerations of the next section show that in the massless case conditions (2.64)–(2.67) are necessary and sufficient in order to be able to define the formal projections P_i, $i = c, a, T$ as bona fide Hilbert space projection operators although they contain the d'alembertian in the denominator. We will see in the next section that these projections are even disjoint after factorizing the zero vectors. It means that the restrictive conditions (2.64)–(2.67) make possible the decomposition of the (restricted) space of (regular) supersymmetric functions into the chiral, antichiral and transversal sectors. Such a decomposition is not possible (for $m = 0$) in general. In fact a similar situation also appears in usual quantum field theory at the point one wants to look at the (quantum) massless vector field as the massless limit of the massive vector field. This is a delicate limit which in fact does not exist. It is related to the Wigner representation theory of the Poincaré group for the massive and massless cases. But there is a way to enforce this limit by the 'method of projections' (for some details see [7], pp 120–121). One introduces longitudinal and transversal projections in the massive case. These projections contain the d'alembertian in the denominator (more precisely the square root of it) and are mathematically not defined in the massless case. The trick in order to define the massless limit (which in fact goes back to Wigner and Bargman) is to do a double factorization of the test function space on which we decide to define our fields as operator valued distributions. The first factorization is given by the divergence condition above and the second one is a zero-vector factorization. In this way a Krein–Hilbert structure appears in both cases, massive and massless, and is exactly the structure which is formally induced in the process of the Gupta–Bleuler quantization by the subsidiary condition (in the frame of the family of gauges mentioned above). The supersymmetric situation is analogue: the role of the divergence condition is now played by a bunch of conditions given in (2.64)–(2.67) and in fact we have found here the rigorous way to perform the method of projections in the supersymmetric case (see [7] for formal considerations in which the authors are not disturbed by the d'alembertian in the denominator; they have good reasons not to be). Certainly the symmetry group is now the supersymmetric one and it is clear that we have touched here its representation theory in the massive and massless cases. Usually the representation theory of the super Poincaré group is performed by using the Clifford structure of the anticommuting translations together with the usual (Abelian) small group in momentum space connected to the usual translations. Considerations above suggest the existence of an alternative Wigner–Makey representation theory of the super Poincaré group in which the (unitary, ireductible) representations are semi-directly 'induced' from a (non-Abelian, supersymmetric) small group related to both anticommuting and commuting translations. This idea is certainly not new and it could be implemented by using either the usual or the 'supersymmetric' Fourier transform (for some useful ideas see [8]); the fact that the supersymmetrization of the Fourier transform is not necessary has been observed in computational work on supersymmetric Feyman integrals).

Before ending our motivation of relations (2.64)–(2.67) from a technical point of view, let us ask ourselves: what else can we do in order to get rid of the unpleasant condition (2.48)? An idea would be to follow the elegant C*-algebra track using the supersymmetric positivity which we have put forward in this paper. This would apparently have the advantage of considering interacting fields from the beginning but will run into known computational weakness of the method. In our opinion the conditions (2.64)–(2.67) are both natural and reasonable enough in order to start at the level of the free fields going to the interacting theory by an operatorial method of the Epstein–Glaser type [10]. They include the usual divergence condition.

Let us come back to the physical interpretation of relations (2.64)–(2.67). On the basis of the discussion above it is clear that they are related to the gauge invariance of the supersymmetric vector field. Indeed they can be considered as a preamble which allows us defining a 'gentle' family of supersymmetric gauges for the supersymmetric vector field. This goes in analogy to the usual case in which the divergence condition can be interpreted as above as a preamble for defining the α-dependent family of gauges induced in the Sueckelberg formalism (Feynman gauge is one of them). It remains to describe explicitly this family of gauges. Because in this discussion we are at the level of free fields the gauges under considerations will be reflected in the corresponding propagators. Going formally from propagators to two-point functions they will be reflected in the two-point functions too. On the other hand, the two-point functions give full information on the inner product of the corresponding Krein structure. It follows that giving a (gentle) family of inner products together with the corresponding Krein structure allows us to specify a family of gauges. Inside this family we can look for those gauges which are compatible with positivity and produce finally what we want: gauges compatible with a (positive) Hilbert space structure. The family of inner products generating the family of supersymmetric gauges will be given in the next section.

A last remark: it can be shown that conditions (2.64)–(2.67) are equivalent to the 'supersymmetric subsidiary conditions'

$$D^2 V^{(-)}(z) = \bar{D}^2 V^{(-)}(z) = 0$$

in a properly defined supersymmetric Fock space [11] in perfect analogy to the usual Gupta–Bleuler case which was shortly explained above. Here $V = V(z)$ is the supersymmetric massless vector field.

The problem of possible zero vectors for the non-negative inner products induced by the kernels $K_i, i = c, a, T$ will be discussed in the next section. For the moment note that there are plenty of them in each sector from the adjacent ones. The 'massless' case in which the measure is $\mathrm{d}\rho(p) = \theta(-p_0)\delta^2(p^2)$ i.e. it is concentrated on the light cone deserves special attention. By putting together the non-negative inner products $(.,.)_i, i = c, a, T$ all zero vectors simply disappear (see section 3). We will construct the natural unique supersymmetric positive definite scalar product and obtain in the next section our rigorous Hilbert–Krein decomposition of the set of supersymmetric functions where conditions (2.64)–(2.67) will play a central role.

3. Hilbert–Krein superspace

In this section we present, on the basis of the results of section 2, the generic Krein structure of supersymmetries. Let V be an inner product space with inner product $\langle .,. \rangle$ and ω an operator on V with $\omega^2 = 1$ (do not confuse this ω with the one in (2.67)). If $(\phi, \psi) = \langle \phi, \omega\psi \rangle; \phi, \psi \in V$ is a (positive definite) scalar product on V then we say that V has a Krein structure. By completing in the scalar product $(.,.)$ we obtain an associated Hilbert space structure (if $(.,.)$ has zero vectors we have in addition to factorize them before completing). We obtain what we call a Hilbert–Krein space (or Hilbert–Krein structure). Hilbert–Krein structures naturally appear in gauge theories (including the well-understood case of electrodynamics; see for instance the book [9]).

Suppose condition (2.48) on the Lorentz invariant measure $\mathrm{d}\rho(p)$ is satisfied and, as always, X and Y are concentrated on its support. We decompose $X = X_1 + X_2 + X_3$ where $X_1 = X_c = P_c X$, $X_2 = X_a = P_a X$, $X_3 = X_T = P_T X$. Then the simplest supersymmetric Hilbert–Krein structure which emerges from the considerations of the preceding section is given by

$$\langle X, Y\rangle = \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_2 \bar{X}^T(z_1) K_0(z_1 - z_2) Y(z_2) \tag{3.1}$$

in the notation

$$X^T = (X_1, X_2, X_3), \qquad Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_T \end{pmatrix}, \qquad K_0(z) = K_0(z) I_3.$$

Here I_3 is the 3×3 identity matrix and X^T is the transpose of X.

Now let

$$(X, Y) = \langle X, \omega Y\rangle \tag{3.2}$$

with

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Certainly $(.,.)$ is positive definite on the basis of results obtained in section 2. It is clear that although each inner product $(.,,)_i$ has zero vectors this will no longer be the case for (3.2).

Although very general the scalar product (3.4) is obstructed by the (from the point of view of applications) unnatural restriction (2.48) of the Lorentz invariant measure $\mathrm{d}\rho$. It holds for the massive but fails for the massless case. Now the restrictions (2.62)–(2.67) on (test) supersymmetric functions come into play. Indeed, under these conditions we can always decompose a supersymmetric function into its chiral, antichiral and transversal parts and write the indefinite as well as the definite scalar products (3.1) and (3.2). Note that in the massless case there is an overlap between chiral/antichiral and transversal sectors which consists of zero vectors and has to be factorized. From (2.11)–(2.13) it follows that a function X belongs to this overlap if

$$X(z) = f(x) + \theta\varphi(x) + \bar{\theta}\bar{\chi}(x) \pm \mathrm{i}\theta\sigma_l\bar{\theta}\partial_l f(x)$$

with

$$\partial_l \varphi \sigma^l = \sigma^l \partial_l \bar{\chi} = 0, \qquad \Box f = 0.$$

The restrictive condition on the measure was transferred to a restrictive condition on (test) functions, a procedure which is common for rigorous quantum gauge fields (see [5, 9]). The Hilbert–Krein structure on supersymmetric functions subjected or not to the conditions (2.64)–(2.67) is the main result of this paper.

We believe that it is justified to call standard Hilbert–Krein supersymmetric space the space of supersymmetric functions with indefinite and (positive) definite inner products given as above by

$$\langle X, Y\rangle = \int \bar{X}^T K_0 Y, \qquad (X, Y) = \langle X, \omega Y\rangle. \tag{3.3}$$

It is exactly the supersymmetric analogue of the relativistic Hilbert space used in quantum field theory in order to produce the Fock space of the free theory [6].

Let us remark that it is possible to generalize the inner products above to

$$(X, Y)_\omega = \langle X, \omega Y\rangle$$

with

$$\omega = \begin{pmatrix} \lambda_c & 0 & 0 \\ 0 & \lambda_a & 0 \\ 0 & 0 & \lambda_T \end{pmatrix}$$

where $\lambda_i, i = c, a, T$ are constants. These inner products are generally indefinite. Let us consider only ω with $\lambda_c = \lambda_a = \lambda$. The supersymmetric kernel associated with the inner product $(.,.)_\omega$ is

$$(\lambda(P_c + P_a) + \lambda_T P_T) K_0(z_1 - z_2).$$

It can be looked at as the two-point function of a quantized free supersymmetric field (the vector one). The corresponding formal propagators are

$$\left[\frac{\lambda_T}{-\Box + m^2} P_T + \frac{\lambda}{-\Box + m^2}(P_c + P_a)\right] k(z_1 - z_2)$$

where $k(z)$ was given in section 2. If the conditions (2.64)–(2.67) are satisfied then we can take the massless limit because the projections are well-defined operators for $m = 0$. We recognize here the propagators of the vector field computed by path integral methods (with a slightly different normalization of λ) given in [1] p 73. The family of gauges is given by the (Stueckelberg type) parameters λ and λ_T. For $\lambda = \lambda_T = -1$ (and $m = 0$) we get the usual propagator of the supersymmetric vector field. It is, as expected, not compatible with positivity. For $\lambda = 0, \lambda_T = -1$ we get a unitary gauge compatible with positivity.

We pass now to some physical applications of the material discussed in this paper. As a first application we mention here that the free chiral/antichiral supersymmetric quantum field theory (i.e. the quantum field formally generated by the free part of the Wess–Zumino Lagrangian) is characterized by the positive definite (at this stage only non-negative) two-point function

$$\begin{pmatrix} \frac{1}{16}\bar{D}^2 D^2 & \frac{m}{4}\bar{D}^2 \\ \frac{m}{4}D^2 & \frac{1}{16}D^2\bar{D}^2 \end{pmatrix} K_0 \tag{3.4}$$

where $\mathrm{d}\rho(p) = \theta(-p_0)\delta(p^2 + m^2)\,\mathrm{d}p$ with $m > 0$. The correspondence to the two-point functions of the chiral Φ and antichiral $\bar{\Phi}$-quantum fields is indicated below,

$$\begin{pmatrix} \Phi\bar{\Phi} & \Phi\Phi \\ \bar{\Phi}\bar{\Phi} & \bar{\Phi}\Phi \end{pmatrix} \sim \begin{pmatrix} \frac{1}{16}\bar{D}^2 D^2 & \frac{m}{4}\bar{D}^2 \\ \frac{m}{4}D^2 & \frac{1}{16}D^2\bar{D}^2 \end{pmatrix} K_0. \tag{3.5}$$

The proof of non-negativity of (3.4) is by computation [11]. The factorization of the zero vectors in (3.4) can be made explicit by imposing the equations of motion $\bar{D}^2\Phi = 4m\Phi$, $D^2\Phi = 4m\bar{\Phi}$ on the test functions [11].

The supersymmetric vacuum coincide with the function one and the supersymmetric Fock space is symmetric (note that following our reasoning all supersymmetric Fock spaces must be symmetric; we expect antisymmetric Fock spaces for ghost fields).

As a second, more interesting application, we shortly describe the (supersymmetric) Epstein–Glaser renormalization method [10] for the massive, interactive supersymmetric Wess–Zumino model. The Epstein–Glaser method, called also causal perturbation theory, is a renormalization method equivalent to the difficult BPHZ renormalization. Its main input is causality in local quantum field theory. It is the only perturbative approach in which renormalization is explored rigorously at the operator level and in which unitarity of the scattering operator is proved (at the level of formal power series). The mathematical tools of the method are the free field (non-interacting) Hilbert space and distribution theory. It works well for massive theories whereas in the massless case some problems with the adiabatic limit appear which can be traced back to difficulties of defining the S-matrix and asymptotic states in this case. It is clear that the present paper offers the tools for performing the supersymmetric causal perturbation theory for the massive interactive Wess–Zumino model because it gives the free Hilbet space structure in which this operator method has to be developed. A first result is a new proof of renormalizability of the massive Wess–Zumino model with $(\Phi^3 + \bar{\Phi}^3)$-interaction [11] including unitarity of the scattering operator.

A non-interacting quantum (free) system consisting of a chiral/antichiral and a (massive) vector part is characterized by the positive definite operator in the standard Hilbert–Krein space (remember $T = -8\Box P_T = D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot{\alpha}} D^2 D^{\dot{\alpha}}$)

$$\begin{pmatrix} \frac{1}{16}\bar{D}^2 D^2 & \frac{m}{4}\bar{D}^2 & 0 \\ \frac{m}{4}D^2 & \frac{1}{16}D^2\bar{D}^2 & 0 \\ 0 & 0 & \frac{1}{8}T \end{pmatrix} K_0. \tag{3.6}$$

Other applications include a supersymmetric Källen–Lehmann representation for interacting Lorentz-scalar supersymmetric quantum fields [11].

Concerning the applications of the present paper one should mention that at the free field level we succeeded to uncover the Hilbert–Krein structure of both massive and massless theories. In the massless case this structure was obtained by restricting the set of allowed test functions, a method which is reminiscent of the Gupta–Bleuler quantization, as was explained in sections 2 and 3. As far as interacting fields are concerned we have strong indications [11] that the operator approach of the causal perturbation theory (Epstein–Glaser method) works well in the massive case whereas massless fields raise problems of similar nature as in the usual quantum field theory; problems which are related to infrared divergences and to the fact that the S-matrix and the asymptotic states are not well defined. Nevertheless, the case of the supersymmetric Abelian gauge theory seems to be still tractable by means of free ghost fields which were already introduced in [11] (for the usual case see [12]). In the non-Abelian case in which ghosts are no longer free the situation seems to be more complicated.

Before ending let us make two remarks. The first concerns the perspective of the present work. We succeeded to uncover the inherent Hilbert–Krein structure of the $N = 1$ superspace. It means that the formal decomposition of supersymmetric functions into chiral, antichiral and transversal components, which was common tool from the first days of superspace, was turned here into what we call the Hilbert–Krein structure of the $N = 1$ superspace or the standard supersymmetric Hilbert–Krein space. It shows that positivity (and as such unitarity) requires the subtraction of the transversal part instead of its addition as this might be suggested by the above-mentioned formal decomposition. Problems with the d'alembertian in the denominator of the projections $P_i, i = c, a, T$ have been discussed. The natural way to avoid singularities is to impose some restrictions on the (test) functions. There are other applications in sight to which we hope to come to (for some first modest steps see [11]).

The second remark is of a technical nature. We worked in the frame of the van der Waerden calculus using Weyl spinors. This is very rewarding from the point of view of computations in supersymmetry but is not totally satisfactory from the rigorous point of view. Indeed, the components of the Weyl spinors as coefficient functions for our supersymmetric (test) functions are supposed to anticommute and this is unpleasant when tracing back the supersymmetric integrals to usual L^2-integrals. Of course this is not a problem. A reformulation of the results using anticommuting Grassmann variables but commuting fermionic components is possible [11]. The net results remain unchanged as they should be.

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